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Analytic perturbation theory in analyzing some QCD observables

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Abstract. This paper is devoted to the application of the recently devised ghost-free analytic perturbation theory (APT) for the analysis of some QCD observables. We start with a discussion of the main problem of the perturbative QCD, ghost singularities, and with a resume of its resolving within the APT. By a few examples in various energy and momentum transfer regions (with the flavor number $f = 3, 4$ and 5) we demonstrate the effect of the improved convergence of the APT modified perturbative QCD expansion. Our first observation is that in the APT analysis the three-loop contribution ($\sim \alpha_s^3$) is as a rule numerically inessential. This gives hope for a practical solution of the well-known problem of the asymptotic nature of the common QFT perturbation series. The second result is that the usual perturbative analysis of time-like events with the large π^2 term in the α_s^3 coefficient is not adequate at $s \leq 2\,\mathrm{GeV}^2$. In particular, this relates to τ decay. Then for the "high" ($f = 5$) region it is shown that the common two-loop (NLO, NLLA) perturbation approximation widely used there (at $10\,\mathrm{GeV} \lesssim s^{1/2} \lesssim 170\,\mathrm{GeV}$) for the analysis of shape/events data contains a systematic negative error at the 1–2 per cent level for the extracted $\bar{\alpha}_s^{(2)}$ values. Our physical conclusion is that the $\bar{\alpha}_s(M_Z^2)$ value averaged over the $f = 5$ data appreciably differs, $\langle\bar{\alpha}_s(M_Z^2)\rangle_{f=5} \simeq 0.124$, from the currently accepted "world average" ($= 0.118$).

1 Preamble

In QCD, a dominant means for the theoretical analysis is based on the perturbation power expansion supported by an appropriate renormalization group (RG) summation. This perturbative QCD (pQCD) satisfactorily correlates the bulk of the experimental data in spite of the fact that the RG invariant power expansion parameter $\bar{\alpha}_s$ is not "small enough" a quantity. Nowadays, the physically accessible region corresponds to three, four and five ($f = 3, 4, 5$) flavor numbers (of active quarks). Just in the three-flavor region there lie unphysical singularities of the central theoretical object, the invariant effective coupling $\bar{\alpha}_s$. These singularities, associated with the QCD scale parameter $\Lambda_{f=3} \simeq 400\,\mathrm{MeV}$, complicate the theoretical interpretation of data in the "small energy" and "small momentum transfer" regions ($s^{1/2}, q \equiv (q^2)^{1/2} \lesssim 1 \div 1.5\,\mathrm{GeV}$). On the other hand, as is well known, their existence contradicts some general statements of the local QFT.

In this paper, we first discuss this main problem of pQCD, the singularities lying in the physically accessible domain, and then give a resume of its solution within the recently devised ghost-free analytic perturbation theory (APT) that resolves the problem without using any additional adjustable parameters. Then we give some impressive results of the application of APT to the analysis of QCD observables.

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1.1 Invariant QCD coupling and observables

Usually, the perturbative QCD part of the theoretical contribution to observables in both the space- and time-like channels is presented in the form of a two- or three-term power expansion

$$\begin{aligned}\frac{O(x)}{O_0} &= 1 + r(x);\\ r(x) &= c_1\bar{\alpha}_s(x) + c_2\bar{\alpha}_s^2 + c_3\bar{\alpha}_s^3 + \ldots;\\ x &= q^2 \text{ or } = s\end{aligned} \tag{1}$$

(our coefficients are normalized by $c_k = C_k\pi^{-k}$, differently from the commonly adopted C_k, like in [1–3]) over powers of the effective QCD coupling $\bar{\alpha}_s$ which is supposed ad hoc to be of the same form as in the two channels, e.g., in the massless three-loop case

$$\begin{aligned}\bar{\alpha}_s^{(3)}(x) &= \frac{1}{\beta_0 L} - \frac{b_1}{\beta_0^2}\frac{\ln L}{L^2}\\ &+ \frac{1}{\beta_0^3 L^3}\left[b_1^2(\ln^2 L - \ln L - 1) + b_2\right]\\ &+ \frac{1}{\beta_0^4 L^4}\left[b_1^3\left(-\ln^3 L + \frac{5}{2}\ln^2 L + 2\ln L - \frac{1}{2}\right)\right.\\ &\left. - 3b_1 b_2 \ln L + \frac{b_3}{2}\right].\end{aligned} \tag{2}$$

Here, $L = \ln(x/\Lambda^2)$, and for the beta-function coefficients we use the normalization

$$\beta(\alpha) = -\beta_0\alpha^2 - \beta_1\alpha^3 - \beta_2\alpha^4 + \ldots = -\beta_0\alpha^2\left(1 + b_1\alpha + b_2\alpha^2 + \ldots\right),$$

that is also free of the π powers. Numerically, they are of the order of unity,

$$\beta_0(f) = \frac{33-2f}{12\pi}; \quad b_1(f) = \frac{153-19f}{2\pi(33-2f)};$$
$$\beta_0(4\pm1) = 0.875 \pm 0.005; \quad b_1(4\pm1) = 0.490^{-0.089}_{+0.076}.$$

Meanwhile, the RG notion of the invariant coupling was first introduced in QED [4] in the space-like region in terms of a real constant z_3 of the finite Dyson renormalization transformation. Just this QED Euclidean invariant charge $\bar{e}(q)$ is the Fourier transform of the space distribution $e(r)$ of the electric charge (arising due to vacuum fluctuations around a "bare" point electron) discussed by Dirac [5] in the thirties – see Appendix IX in the textbook in [6].

Generally, in the RG formalism (for details, see, e.g., the chapter on the renormalization group in the monograph in [7] and/or Sect. 1 in [8]) the notion of the invariant coupling $\bar{g}(q)$ is *defined only in the space-like domain.*

In particular, this means that if some observable $O(q^2)$ is physically a function of one kinematic Lorentz invariant space-like argument q^2, then, due to its renormalization invariance, it should be a function of RG invariants only. For instance, in the one coupling massless case

$$O(q^2/\mu^2, g_\mu) = F\left(\bar{g}(q^2/\mu^2, g_\mu)\right) \quad \text{with} \quad F(g) = O(1,g).$$

Due to this important property, in the weak coupling case we deal with the *functional expansion* of an observable $O(q^2)$ in powers of the invariant coupling $\bar{g}$. This is the real foundation of the QCD power expansion (1) in the Euclidean case with $x = q^2$. At the same time, within the RG formalism there is no natural means for defining the invariant coupling $\tilde{g}(s); s = -q^2$ and the perturbative expansion for an observable $\tilde{O}(s)$ in the time-like region.

Nevertheless, in modern practice, people commonly use the same singular expression for the QCD effective coupling $\bar{\alpha}_s$, like (2), in both the space- and time-like domains. The only price usually paid for this transferring from the Euclidean to the Minkowskian region is a change of the numerical expansion coefficients. The time-like ones $c_{k\geq3} = d_k - \delta_k$ include negative "π^2 terms" proportional to π^2 and lower expansion coefficients c_k,

$$\delta_3 = \frac{(\pi\beta_0(f))^2}{3}c_1, \qquad \delta_4 = (\pi\beta_0)^2\left(c_2 + \frac{5}{6}\,b_1c_1\right)\ldots \qquad (3)$$

These (rather essential, as far as $\pi^2\beta_0^2(f = 4\pm1) = 4.340^{-.666}_{+.723}$) structures δ_k arise [9–12] in the course of analytic continuation from the Euclidean to the Minkowskian region. The coefficients d_k should be treated as genuine kth order ones. They are calculated via the relevant Feynman diagrams.

Table 1. Minkowskian c_k and Euclidean $d_i = c_i + \delta_i$ expansion coefficients and their differences

Process	f	$c_1 = d_1$	$c_2 = d_2$	c_3	d_3	δ_3	δ_4
τ decay	3	$1/\pi$	.526	0.852	1.389	0.537	5.01
e^+e^-	4	.318	.155	−0.351	0.111	0.462	2.451
e^+e^-	5	.318	.143	−0.413	−0.023	0.390	1.752
Z_0 decay	5	.318	.095	−0.483	−0.094	0.390	1.576

To demonstrate the importance of the "π^2 terms", we took the $f = 3$ case for τ decay, the $f = 4,5$ cases for $e^+e^- \to$ hadron annihilation and the Z_0 decay (with $f = 5$); see Table 1, in which we also give values for the π^2 terms. In the normalization (1), all coefficients c_k, d_k and δ_k are of the order of unity. In the $f = 4,5$ region the contribution δ_3 prevails in c_3 and $|d_3| \ll |c_3|$ (see also Table II in Bjorken's review [11]).

1.2 Unphysical singularities

Let us remind the reader that the ghost-trouble first discovered in QED in the mid-fifties (and quite soon in the renormalizable version of the pion–nucleon interaction) was considered there as a serious argument in favor of the inner inconsistency of the whole local QFT. In the QED case, the ghost singularity lies far above the mass of the Universe and has no pragmatic meaning.

However, in QCD it lies in the quite physical infrared (IR) region and we are forced to face it. This means that, if one believes in QCD as a consistent, physically important theory, one has no other possibility than to consider the QCD unphysical singularities as an artefact of some approximations used in pQCD. This point of view is supported by some lattice simulations and by the solution of the Schwinger–Dyson equations – see, e.g., Sect. 5.3. in the recent review in [13].

For illustration of the fundamental inconsistency of the current pQCD practice connected with unphysical singularities, take the well-known relation between the so-called Adler function D and the total cross-section ratio R of the related process

$$D(q^2) = q^2\int_0^\infty \frac{R(s)\mathrm{d}s}{(s+q^2)^2}. \qquad (4)$$

In the case of inclusive e^+e^- annihilation into hadrons, $R(s)$ is the ratio of cross-sections presented in the form $R(s) = 1 + r(s)$ with the function r expandable in powers of $\bar{\alpha}_s(s)$ like in (1). At the same time, the Adler function is also used in the form $D = 1 + d$ with d expanded in powers of $\bar{\alpha}_s(q^2)$.

Here, we face two paradoxes. First, $\bar{\alpha}_s(q^2)$ and, hence, the perturbative $D(q^2)$ obeys – see (2) – a non-physical singularity at $q^2 = \Lambda^2$ in evident contradiction with the representation (4). Second, the integrand $R(s)$, being expressed via powers of $\bar{\alpha}_s(s)$, obeys non-integrable singularities at $s = \Lambda^2$, which makes the r.h.s. of (4) senseless.

This second problem is typical of inclusive cross-sections, e.g., for the hadronic τ decay. Generally, in the current literature it is treated in a very strange way: by shifting the contour of integration from the real axis with strong singularities on it into the complex plane. However, such a "physical" trick cannot be justified within the theory of complex variables.

1.3 The "ghost" problem resolving

Meanwhile, as is known from the early eighties, the perturbation representation (1) for the Minkowskian observable with the coefficients modified by the π^2 terms is valid only at small parameter $\pi^2/\ln^2(s/\Lambda^2)$ values; that is, in the region of sufficiently high energies $W \equiv s^{1/2} \gg \Lambda e^{\pi/2} \simeq 2\,\mathrm{GeV}$.

Here, it is appropriate to recall the construction devised by Radyushkin [9] and Krasnikov–Pivovarov [10] (RKP procedure) about twenty years ago. These authors used the integral transformation

$$R(s) = \frac{\mathrm{i}}{2\pi}\int_{s-\mathrm{i}\varepsilon}^{s+\mathrm{i}\varepsilon}\frac{\mathrm{d}z}{z}D_{\mathrm{pt}}(-z) \equiv \mathbf{R}\left[D(q^2)\right], \tag{5}$$

reverse to the Adler relation (4) (that is treated now as an integral transformation)

$$R(s) \to D(q^2) = q^2\int_0^\infty \frac{R(s)\mathrm{d}s}{(s+q^2)^2} \equiv \mathbf{D}\left\{R(s)\right\} \tag{6}$$

for the defining modified expansion functions

$$\mathfrak{A}_k(s) = \mathbf{R}[\alpha_{\mathrm{s}}^k(q^2)] \tag{7}$$

for the perturbative QCD contribution

$$r(s) = d_1\mathfrak{A}_1(s) + d_2\mathfrak{A}_2(s) + d_3\mathfrak{A}_3(s) \tag{8}$$

to an observable in the time-like region.

At the one-loop level, with the effective coupling $\bar{\alpha}_{\mathrm{s}}^{(1)} = \left[\beta_0\ln(q^2/\Lambda^2)\right]^{-1}$ one has

$$\begin{aligned}\mathfrak{A}_1^{(1)}(s) = \mathbf{R}\left[\bar{\alpha}_{\mathrm{s}}^{(1)}\right] &= \frac{1}{\pi\beta_0}\arccos\frac{L}{\sqrt{L^2+\pi^2}}\\ &= \frac{1}{\beta_0}\left[\frac{1}{2}-\frac{1}{\pi}\arctan\frac{L}{\pi}\right];\\ L = \ln\frac{s}{\Lambda^2},\end{aligned} \tag{9}$$

and for higher functions

$$\begin{aligned}\mathfrak{A}_2^{(1)}(s) &= \frac{1}{\beta_0^2\left[L^2+\pi^2\right]}; \quad \mathfrak{A}_3^{(1)}(s) = \frac{L}{\beta_0^3\left[L^2+\pi^2\right]^2};\\ \mathfrak{A}_4^{(1)}(s) &= \frac{L^2-\pi^2/3}{\beta_0^4\left[L^2+\pi^2\right]^3},\end{aligned} \tag{10}$$

which *are not powers* of $\mathfrak{A}_1^{(1)}(s)$.

The r.h.s. of (9) at $L \geq 0$ can also be presented in the form

$$\mathfrak{A}_1^{(1)}(s) = \frac{1}{\pi\beta_0}\arctan\frac{\pi}{L}, \tag{9a}$$

convenient for the UV analysis. Just this form, (9a), was discovered in the early eighties in [14,9], while (10) was found in [9,10]. All these papers dealt with HE behavior and did not pay proper attention to the region $L \leq 0$.

On the other hand, expression (9) was first discussed only 15 years later by Milton and Solovtsov [15]. It were these authors who first made the important observation that expression (9) represents a continuous monotonical function without an unphysical singularity at $L = 0$ and proposed to use it as an effective "Minkowskian QCD coupling" $\tilde{\alpha}(s) \equiv \mathfrak{A}_1(s)$ in the time-like region.

For the two-loop case, to the popular approximation

$$\beta_0\bar{\alpha}_{\mathrm{s,pop}}^{(2)}(q^2) = \frac{1}{l} - b_1(f)\frac{\ln l}{l^2}; \quad l = \ln\frac{q^2}{\Lambda^2}$$

there corresponds [9,16]

$$\begin{aligned}\tilde{\alpha}_{\mathrm{pop}}^{(2)}(s) \equiv \mathfrak{A}_1^{(2,\mathrm{pop})}(s) &= \left(1+\frac{b_1 L}{L^2+\pi^2}\right)\tilde{\alpha}^{(1)}(s)\\ &\quad - \frac{b_1}{\beta_0}\frac{\ln\left[\sqrt{L^2+\pi^2}\right]+1}{L^2+\pi^2}.\end{aligned} \tag{11}$$

For $L \gg \pi$, by expanding this expression and $\mathfrak{A}_2$ from (10) in powers of π^2/L^2 we arrive at the π^2 terms (3).

Both the functions (9) and (11) are monotonically decreasing with the finite IR value $\tilde{\alpha}(0) = 1/\beta_0(f=3) \simeq 1.4$. They have no singularity at $L = 0$. Higher functions go to zero, $\mathfrak{A}_k(0) = 0$, in the IR limit.

As has first been noticed in [17,18], by applying the transformation $\mathbf{D}$ in (6) to the functions $\mathfrak{A}_k(s)$, instead of $\bar{\alpha}_{\mathrm{s}}(q^2)$ powers, we obtain expressions $\mathbf{D}[\mathfrak{A}_k(s)] = \mathcal{A}_k(q^2)$ that are also free of unphysical singularities. These functions have been discussed in the nineties [19–24] in the context of the so-called "analytic approach" to perturbative QCD.

Therefore, this analytic approach in the Euclidean region and the RKP formulation for Minkowskian observables can be united in a single scheme, the "analytic perturbation theory", APT, that has been formulated quite recently in our papers of [17,18]. In the next section, we give a short resume of this APT construction and then, in Sects. 3 and 4, we present the results of its practical applications.

2 The APT, a closed theoretical scheme

The APT scheme closely relates two ghost-free formulations of the modified perturbation expansion for observables.

2.1 Relation between Euclidean and Minkowskian functions

The first one, that was initiated in the early eighties [9,10] and outlined above, changes the standard power expansion (1) in the time-like region into the non-power one of (8). It uses the operation in (5), that is reverse, $\mathbf{R} = [\mathbf{D}]^{-1}$, to the one defined by the "Adler relation" (6) and transforms a real function $R(s)$ of a positive (time-like) argument into a real function $D(q^2)$ of a positive (space-like) argument.

By the operation $\mathbf{R}$, one can define [15] the RG invariant Minkowskian coupling $\tilde{\alpha}(s) = \mathbf{R}\,[\bar{\alpha}_{\rm s}]$, and its "effective powers" (7) that are free of ghost singularities. Some examples are given by (9), (10) and (11). At the one-loop level, they are related by the differential recursion relation $k\beta_0\mathfrak{A}^{(1)}_{k+1} = -(\mathrm{d}/\mathrm{d}L)\mathfrak{A}^{(1)}_k$ and *are not powers* of $\mathfrak{A}^{(1)}_1$.

By applying $\mathbf{D}$ to $\mathfrak{A}_k(s)$, one can "try to return" to the Euclidean domain. However, instead of $\alpha_{\rm s}$ powers, we arrive at some other functions,

$$\mathcal{A}_k(q^2) = \mathbf{D}\,[\mathfrak{A}_k]\,, \tag{12}$$

analytic in the cut q^2-plane and free of ghost singularities. In the one-loop case,

$$\beta_0\mathcal{A}^{(1)}_1(q^2) = \frac{1}{\ln(q^2/\Lambda^2)} - \frac{\Lambda^2}{q^2-\Lambda^2},$$
$$\beta_0^2\mathcal{A}^{(1)}_2(q^2) = \frac{1}{\ln^2(q^2/\Lambda^2)} + \frac{q^2\Lambda^2}{(q^2-\Lambda^2)^2}, \dots \tag{13}$$

These expressions have originally been obtained by other means [19,20] in the mid-nineties. The first function $\mathcal{A}_1 = \alpha_{\rm an}(q^2)$, an analytic invariant Euclidean coupling, should now be treated as a *counterpart* of the invariant Minkowskian coupling $\tilde{\alpha}(s) = \mathfrak{A}_1(s)$. Both $\alpha_{\rm an}$ and $\tilde{\alpha}$ are real monotonically decreasing functions with the same maximum value

$$\alpha_{\rm an}(0) = \tilde{\alpha}(0) = 1/\beta_0(f=3) \simeq 1.4$$

in the IR limit. (Note that the transition from the usual invariant $\overline{\rm MS}$ coupling $\alpha_{\rm s}$ to the Minkowskian $\tilde{\alpha}$ and Euclidean $\alpha_{\rm an}$ ones can be understood as a transformation to new renormalization schemes. In the one-loop case,

$$\alpha_{\rm s} \to \tilde{\alpha}^{(1)} = \frac{1}{\pi\beta_0}\arctan(\pi\beta_0\alpha_{\rm s})$$

and

$$\alpha_{\rm s} \to {\alpha_{\rm an}}^{(1)} = \alpha_{\rm s} + \frac{1}{\beta_0}\left(1 - \mathrm{e}^{1/\beta_0\alpha_{\rm s}}\right)^{-1}. \tag{14}$$

Here, the first transition looks "quite usual" as $\tilde{\alpha}$ can be expanded in powers of $\alpha_{\rm s}$, while the second one in the weak coupling case behaves like the identity transformation as far as the second non-perturbative term $\mathrm{e}^{-1/\beta_0\alpha_{\rm s}}$ leaves no "footsteps" in the power expansion. For both $\tilde{\alpha}^{(1)}$ and ${\alpha_{\rm an}}^{(1)}$, the corresponding β-functions have a zero at $\alpha = 1/\beta_0$ and are symmetric under reflection, $[\alpha - 1/2\beta_0] \to -[\alpha - 1/2\beta_0]$. Moreover, the β function for $\tilde{\alpha}(s)$ turns out to be equal to the spectral function for $\alpha_{\rm an}(q^2)$ – see below (18) at $k=1$.) All higher functions vanish, $\mathcal{A}_k(0) = \mathfrak{A}_k(0) = 0$, in this limit. For $k \geq 2$, they oscillate in the IR region and form [25,26] an asymptotic sequence à lá Erdélyi.

The same properties remain valid for the higher-loop case. Explicit expressions for $\mathcal{A}_k$ and $\mathfrak{A}_k$ at the two-loop case can be written down [27,28]) in terms of a special Lambert function. They are illustrated below in Figs. 1a,b. Note here that to relate Euclidean and Minkowskian functions, instead of integral expressions (5) and (6) one can use simpler relations, in terms of spectral functions $\rho(\sigma) = \Im\mathcal{A}(-\sigma)$,

$$\mathcal{A}_k(q^2;f) = \frac{1}{\pi}\int_0^\infty \frac{\mathrm{d}\sigma}{\sigma+q^2}\,\rho_k(\sigma;f);$$
$$\mathfrak{A}_k(s;f) = \frac{1}{\pi}\int_s^\infty \frac{\mathrm{d}\sigma}{\sigma}\rho_k(\sigma;f), \tag{15}$$

equivalent to the expressions $\mathcal{A}_k(q^2) = \mathbf{D}\,[\mathfrak{A}_k]$, and $\mathfrak{A}_k(s) = \mathbf{R}[\mathcal{A}_k]$.

Remarkably enough, the mechanism of the liberation of unphysical singularities is quite different. While in the space-like domain it involves non-perturbative structures in powers of q^2, in the time-like region it is based only on a resummation of the "π^2 terms". Figuratively, (non-perturbative!) *analyticization* [19,20,26] in the q^2-channel can be treated as a quantitatively distorted reflection (under $q^2 \to s = -q^2$) of a (perfectly perturbative) π^2 resummation in the s-channel. This "distorting mirror" effect, first discussed in [15,29], is clearly seen in Figs. 1a,b mentioned above.

This means also that the introduction of non-perturbative $1/q^2$ structures now has got *another motivation, see (12), independent of the analyticization prescription.*

2.2 Global APT

In reality, a physical domain includes regions with various "numbers of active quarks", i.e., diverse flavor numbers, $f = 3, 4, 5$ and 6. In each of these regions, we deal with a different amount of quark quantum fields; that is, with distinct QFT models, with corresponding Lagrangians. To combine them into a joint picture, the procedure of the threshold matching is to be used. It establishes relations between renormalization procedures for a model with different f values.

For example, in the $\overline{\rm MS}$ scheme the matching relation has a simple form:

$$\bar{\alpha}_{\rm s}(q^2 = M_f^2; f-1) = \bar{\alpha}_{\rm s}(q^2 = M_f^2; f). \tag{16}$$

It defines a "global effective coupling"

$$\bar{\alpha}_{\rm s}(q^2) = \bar{\alpha}_{\rm s}(q^2;f) \quad \text{at } M_{f-1}^2 \leq q^2 \leq M_f^2,$$

continuous in the space-like region of positive q^2 values with a discontinuity of the derivatives at the matching

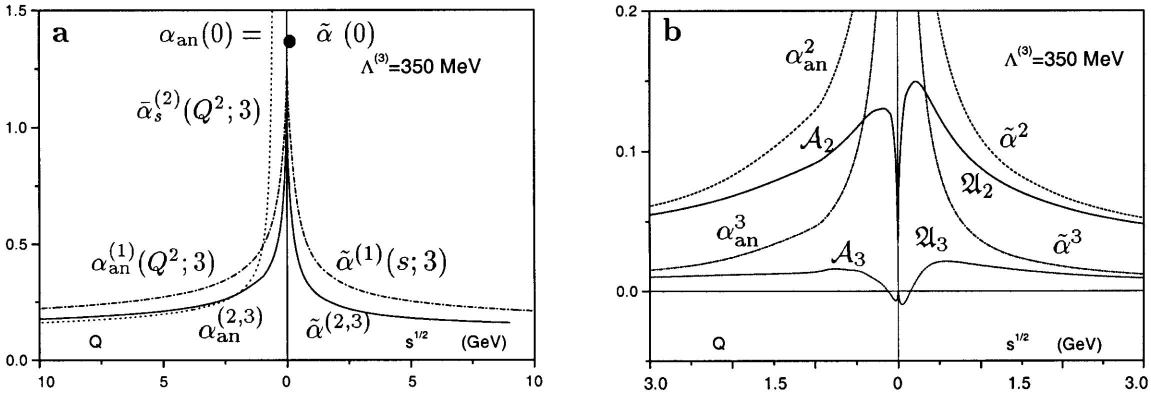

Fig. 1. a Space-like and time-like global analytic couplings in the domain of a few GeV with $f = 3$ and $\Lambda^{(3)} = 350\,\mathrm{MeV}$; **b** "distorted mirror symmetry" for global expansion functions. All the curves in (b) correspond to exact two-loop solutions expressed in terms of the Lambert function

points $q^2 = M_f^2$. To this global $\bar{\alpha}_{\rm s}$, there corresponds a discontinuous spectral density

$$\rho_k(\sigma) = \rho_k(\sigma;3) + \sum_{f\geq 4} \theta(\sigma - M_f^2)\,\{\rho_k(\sigma;f) - \rho_k(\sigma;f-1)\}\,, \tag{17}$$

with $\rho_k(\sigma;f) = \Im\bar{\alpha}_{\rm s}^k(-\sigma,f)$ which yields [17,18] via relations analogous to (15)

$$\mathcal{A}_k(q^2) = \frac{1}{\pi}\int_0^\infty \frac{d\sigma}{\sigma+q^2}\,\rho_k(\sigma); \quad \mathfrak{A}_k(s) = \frac{1}{\pi}\int_s^\infty \frac{d\sigma}{\sigma}\rho_k(\sigma), \tag{18}$$

the smooth global Euclidean and spline–continuous global Minkowskian expansion functions.

In Fig. 1a, by the dotted line we give the usual two-loop effective QCD coupling $\bar{\alpha}_{\rm s}(q^2)$ with a singularity at $q^2 = \Lambda^2$. Meanwhile, the dash-dotted curves represent the one-loop APT expressions (9) and (13). The solid APT curves are based on the exact two-loop solutions of the RG equations and approximate three-loop solutions in the $\overline{\rm MS}$ scheme. Their remarkable coincidence (within 2–4 per cent) demonstrates the reduced sensitivity of the APT approach (see also [20–22]) with respect to higher-loop effects in the whole Euclidean and Minkowskian regions from the IR to the UV limits. Figure 1b shows higher two-loop functions in comparison with the $\alpha_{\rm an}$ and $\tilde{\alpha}$ powers.

Generally, the functions $\mathfrak{A}_k$ and $\mathcal{A}_k$ differ from the local ones with a fixed f value. Minkowskian global functions $\mathfrak{A}_k$ can be presented via $\mathfrak{A}_k(s,f)$ by the relations

$$\tilde{\alpha}(s) = \tilde{\alpha}(s;f) + c(f); \quad \mathfrak{A}_2(s) = \mathfrak{A}_2(s;f) + \mathfrak{c}_2(f) \quad \text{at } M_f^2 \leq s \leq M_{f+1}^2, \tag{19}$$

with *shift constants* $c(f), \mathfrak{c}_2(f)$ representable via integrals over $\rho_k(\sigma;f+n)$, $n \geq 1$ with additional reservations, like $c(6) = 0$, related to the asymptotic freedom condition.

The numerical estimate performed in [18] (see also Table 6 in [27]) for traditional values of the QCD scale parameter, $\Lambda_3 \sim 300$–$400\,\mathrm{MeV}$,

$$c(3) \sim 0.02, \quad c(4) \simeq 3.10^{-3}, \quad c(5) \simeq 3.10^{-4}; \quad \mathfrak{c}_2(f) \simeq 3\alpha(M_f^2)c(f)$$

reveals that these constants are essential in the $f = 3, 4$ region at a few per cent level for $\tilde{\alpha}$ and at the circa 10% level for $\mathfrak{A}_2$.

Meanwhile, the global Euclidean functions $\mathcal{A}_k(q^2)$ cannot be related to the local ones $\mathcal{A}_k(q^2, f)$ by simple relations. Nevertheless, numerical calculation shows [27,28] that in the $f = 3$ region one has approximately

$$\alpha_{\rm an}(q^2) = \alpha_{\rm an}(q^2;3) + c(3); \quad \mathcal{A}_2(q^2) = \mathcal{A}_2(q^2);3) + \mathfrak{c}_2(3) \quad \text{at } M_3^2 \leq s \leq M_4^2. \tag{20}$$

3 The APT applications

3.1 General comments

In what follows, we abstract from recent successive use of the analytic approach to hadronic formfactors [30] and concentrate on the QCD applications of APT.

To illustrate the quantitative difference between the global APT scheme and common practice of data analysis in perturbative QCD, consider a few examples.

In the usual treatment (see, e.g., [1]) the (QCD perturbative part of) a Minkowskian observable, like the e^+e^- annihilation or Z_0 decay cross-section ratio, is presented in the form

$$R(s) = R_0\,(1 + r(s))\,; \quad r_{\rm PT}(s) = c_1\bar{\alpha}_{\rm s}(s) + c_2\bar{\alpha}_{\rm s}^2(s) + c_3\bar{\alpha}_{\rm s}^3(s) + \ldots \tag{21}$$

Here, the coefficients c_1, c_2 and c_3 are not decreasing numerically; see Table 1. A rather large negative c_3 value comes mainly from the $-c_1\pi^2\beta_0^2/3$ term. In the APT, we have instead

$$r_{\rm APT}(s) = d_1\tilde{\alpha}(s) + d_2\mathfrak{A}_2(s) + d_3\mathfrak{A}_3(s) + \ldots \tag{22}$$

Table 2. Relative contributions (in%) of one-, two- and three-loop terms to observables

Process	q or $s^{1/2}$	f	PT			APT		
GLS sum rule	1.73 GeV	4	65	24	11	75	21	**4**
Bjorken. s.r.	1.73 GeV	3	55	26	19	80	19	**1**
Incl. τ decay	0–2 GeV	3	55	29	16	88	11	**1**
$e^+e^- \to$ hadr.	10 GeV	4	96	8	−4	92	7	**.5**
$Z_o \to$ hadr.	89 GeV	5	98.6	3.7	−2.3	96.9	3.5	**−.4**

with reasonably decreasing Feynman coefficients $d_{1,2} = c_{1,2}$ and $d_3 = c_3 + c_1\pi^2\beta_0^2/3$, the mentioned π^2 term of c_3 being "swallowed" by $\tilde{\alpha}(s)$.

In the Euclidean channel, instead of a power expansion similar to (21), we typically have

$$d_{\mathrm{APT}}(q^2) = d_1\alpha_{\mathrm{an}}(q^2) + d_2\mathcal{A}_2(q^2) + d_3\mathcal{A}_3(q^2) + \ldots \quad (23)$$

with the same coefficients d_k extracted from the Feynman diagrams. Here the modification is related to non-perturbative structures, in powers of q^2, like in (13).

In Table 2, we give values of the relative contribution of the first, second, and third terms of the r.h.s. in (21), (22) and (23) for the Gross–Llywellin-Smith [31] and Bjorken [32] sum rules, τ decay in the vector channel [33], as well as for e^+e^- and Z_0 inclusive cross-sections. As follows from this table, in the APT case, the three-loop (last) term is very small, and on being compared with the data errors, numerically inessential. This means that, in practice, *one can use the APT expansions (22) and (23) without the last term.*

3.2 Semi-quantitative estimate

This conclusion can be valuable for the case when the three-loop contribution, i.e., d_3, is unknown. Here, some use the so-called NLLA approximation, which is common practice in the $f = 5$ region. For the Minkowskian observable, e.g., in the event–shape (see, e.g., [34]) the analysis there corresponds to the two-term expression

$$r(s) = c_1\alpha_{\mathrm{s}}(s) + c_2\alpha_{\mathrm{s}}^2(s). \quad (24)$$

On the basis of the numerical estimates of Table 1, in such a case, we recommend instead *to use the two-term APT representation*

$$r^{(2)}_{\mathrm{APT}}(s) = d_1\tilde{\alpha}(s) + d_2\mathfrak{A}_2(s), \quad (25)$$

which, with $L^2 \gg \pi^2$, is equivalent to the three-term expression

$$r_3^{\Delta}(s) = d_1\left\{\bar{\alpha}_{\mathrm{s}} - \frac{\pi^2\beta_0^2}{3}\bar{\alpha}_{\mathrm{s}}^3\right\} + d_2\bar{\alpha}_{\mathrm{s}}^2 = c_1\bar{\alpha}_{\mathrm{s}} + c_2\bar{\alpha}_{\mathrm{s}}^2 - \boldsymbol{\delta_3\bar{\alpha}_{\mathrm{s}}^3}, \quad (26)$$

i.e., to take into account the known predominant π^2 part of the next coefficient c_3. As follows from a comparison of the last expression with the previous, the two-term one (24), the $\bar{\alpha}_{\mathrm{s}}$ numerical value extracted from (26), for the same measured value r_{obs}, will differ mainly by a positive quantity (e.g., in the $f = 5$ region with $\bar{\alpha}_{\mathrm{s}} \simeq 0.12 \div 0.15$):

$$(\triangle\bar{\alpha}_{\mathrm{s}})_3 = \left.\frac{\pi\delta_3\bar{\alpha}_{\mathrm{s}}^3}{1 + 2\pi d_2\bar{\alpha}_{\mathrm{s}}}\right|^{f=5}_{20\div 100\,\mathrm{GeV}} = \frac{1.225\bar{\alpha}_{\mathrm{s}}^3}{1 + 0.90\bar{\alpha}_{\mathrm{s}}} \simeq 0.002 \div 0.003, \quad (27)$$

which turns out to be numerically important.

Moreover, in the $f = 4$ region, where the three-loop (NNLLA) approximation is commonly used in the data analysis, the π^2 term δ_4 of the next order turns out also to be essential. Hence, we propose there, instead of (21), *to use the APT three-term expression*

$$r^{(3)}_{\mathrm{APT}}(s) = d_1\tilde{\alpha}(s) + d_2\mathfrak{A}_2(s) + d_3\mathfrak{A}_3(s), \quad (28)$$

approximately equivalent to the four-term one

$$r_4^{\Delta}(s) = d_1\bar{\alpha}_{\mathrm{s}} + d_2\bar{\alpha}_{\mathrm{s}}^2 + c_3\bar{\alpha}_{\mathrm{s}}^3 - \boldsymbol{\delta_4\bar{\alpha}_{\mathrm{s}}^4}; \quad c_3 = d_3 - \delta_3, \quad (29)$$

or to

$$r_4^{\Delta}(s) = d_1\left\{\bar{\alpha}_{\mathrm{s}} - \frac{\pi^2\beta_0^2}{3}\bar{\alpha}_{\mathrm{s}}^3 - b_1\frac{5}{6}\pi^2\beta_0^2\bar{\alpha}_{\mathrm{s}}^4\right\} + d_2\left\{\bar{\alpha}_{\mathrm{s}}^2 - \pi^2\beta_0^2\bar{\alpha}_{\mathrm{s}}^4\right\} + d_3\bar{\alpha}_{\mathrm{s}}^3,$$

with δ_3 and δ_4 defined [9,12] in (3).

The three- and two-term structures in braces are related to the specific expansion functions $\tilde{\alpha} = \mathfrak{A}_1$ and $\mathfrak{A}_2$ defined above (18) and entering into the non-power expansion (28).

To roughly estimate the numerical effect of using this last modified expression (29), we take the e^+e^- inclusive annihilation. For $s^{1/2} \simeq 3 \div 5\,\mathrm{GeV}$ with $\bar{\alpha}_{\mathrm{s}} \simeq 0.28 \div 0.22$, one has

$$(\triangle\bar{\alpha}_{\mathrm{s}})_4 = \left.\frac{\pi\delta_4\bar{\alpha}_{\mathrm{s}}^4}{1 + 2\pi d_2\bar{\alpha}_{\mathrm{s}}}\right|^{f=4}_{3\div 5\,\mathrm{GeV}} = \frac{1.07\bar{\alpha}_{\mathrm{s}}^4}{1 + 0.974\bar{\alpha}_{\mathrm{s}}} \simeq 0.005 \div 0.002,$$

which is an important effect on the level of circa $1 \div 2\%$.

Moreover, the $(\triangle\bar{\alpha}_{\mathrm{s}})_4$ correction turns out to be noticeable even in the lower part of the $f = 5$ region! Indeed, to $s^{1/2} \simeq 10 \div 40\,\mathrm{GeV}$ with $\bar{\alpha}_{\mathrm{s}} \simeq 0.20 \div 0.15$ there corresponds

$$(\triangle\bar{\alpha}_{\mathrm{s}})_4\big|^{f=5}_{10\div 40\,\mathrm{GeV}} \simeq 0.71\bar{\alpha}_{\mathrm{s}}^4 \simeq (1.1\div 0.3)\cdot 10^{-3} \quad (\lesssim 0.5\%).$$

3.3 Important warning

It is essential to note that the approximate expressions (26) and (29) are equivalent to the exact ones (25) and (28) only in the region $L = \ln\left(s/\Lambda^2\right) \gg \pi$ as shown on Fig. 3.3.

One can see that the curve for approximate Minkowskian coupling,

$$\tilde{\alpha}_{\mathrm{appr}}(s) = \bar{\alpha}_{\mathrm{s}}(s) - (\pi^2\beta_0^2/3)\bar{\alpha}_{\mathrm{s}}^3, \quad (30)$$

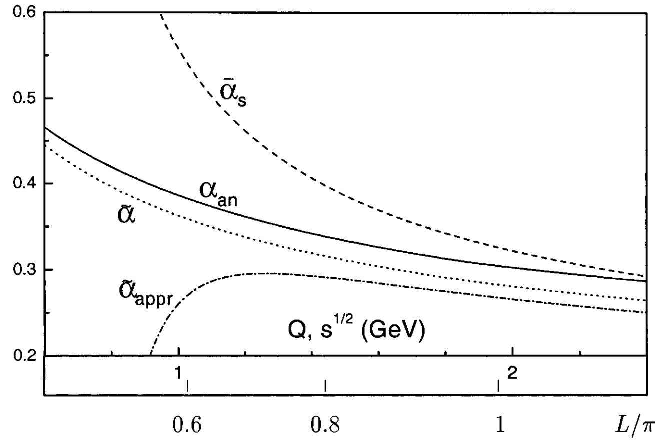

Fig. 2. Comparison of common QCD coupling $\bar{\alpha}_s$ with the APT global ones ($\tilde{\alpha}$, α_{an}) in the $q, s^{1/2} < 3\,\mathrm{GeV}$ region at $\Lambda_3 = 400\,\mathrm{MeV}$. By a dash-dotted line we indicate the approximate Minkowskian coupling (30). All the curves are taken (see Tables 1, 5 and 6 in [28]) for the three-loop global case

which precisely corresponds to the popular approximation (21) (and gives rise to the π^2term in the α_s^3 coefficient) has a rather peculiar behavior. In the region $L > \pi$ it goes rather close to the curve for $\tilde{\alpha}$. For instance, at $L \simeq \pi$ the relative error of the approximation is about 5 per cent. On the other hand, below $L \simeq 0.8\pi$ (i.e., $W \simeq 1.0$–$1.4\,\mathrm{GeV}$) the distance $\tilde{\alpha}$–$\tilde{\alpha}_{appr}$ between the curves (error of approximation) increases and at $L \simeq 0.7\pi$ it blows up (or rather "comes down").

In particular, at $s \leq 2\,\mathrm{GeV}^2$ it is rather inappropriate to refer to $\bar{\alpha}_s(s)$ and it is erroneous to use $\tilde{\alpha}_{appr}(s)$ and the common expansion (21).

This means that *below* $s = 2\,\mathrm{GeV}^2$ *it is inadequate to use the common* $\bar{\alpha}_s(s)$ *and power expansion (21).*

In other words, we claim that below $s = 2\,\mathrm{GeV}^2$ it is an intricate business to analyze data in terms of the "good old" (but singular) α_s [1]. Here, the approximate relation (30) does not work as illustrated in Fig. 3.3.

In this low-energy Minkowskian/Euclidean region data have to be analyzed in terms of the non-power expansion of (22) and (23), and the extracted parameter should be $\alpha_{an}(s), \tilde{\alpha}(q^2)$ or $\Lambda^{(3)}$. In Table 3 we give a few numerical examples for the chain

$$\alpha_{an}(M_\tau) \leftrightarrow \tilde{\alpha}(M_\tau) \leftrightarrow \Lambda^{(3)} \to \Lambda^{(5)} \leftrightarrow \bar{\alpha}_s(M_Z),$$

Table 3. Numerical chain related LE with HE regions

$\tilde{\alpha}(M_\tau)$	$\alpha_{an}(M_\tau)$	$\Lambda^{(3)}$	$\Lambda^{(5)}$	$\bar{\alpha}_s(M_Z)$
0.309	0.332	450 MeV	303 MeV	0.125
0.292	0.314	400 MeV	260 MeV	0.121
0.278	0.299	350 MeV	218 MeV	0.119
0.266	0.286	300 MeV	180 MeV	0.116

which allows one to study the QCD theoretical compatibility of LE data with the HE ones in the APT analysis.

Here, the main element of the correlation is the chain $\Lambda^{(3)} \leftrightarrow \Lambda^{(3)} \leftrightarrow \Lambda^{(5)}$ that follows from the matching condition (16)[2].

4 Quantitative illustration

Consider now a few cases in the $f = 5$ region.

Υ decay. According to the Particle Data Group (PDG) overview (see their Fig. 9.1 on page 88 of [1]), this is (with $\alpha_s(M_\Upsilon^2) \simeq 0.170$ and $\bar{\alpha}_s(M_Z^2) = 0.114$) one of the most "annoying" points of their summary of $\bar{\alpha}_s(M_Z^2)$ values. It is also singled out theoretically. The expression for the ratio of decay widths starts with the cubic term[3]

$$R(\Upsilon) = R_0 \alpha_s^3(\xi M_\Upsilon^2)(1 - e_1 \alpha_s),$$

with $\xi \lesssim 0.5$ and $c_1(\xi) \simeq 1$. Due to this, the π^2 corrections[4] corresponding to the APT expression

$$R_{\mathrm{APT}}(\Upsilon) = R_0 \mathfrak{A}_3(\xi M_\Upsilon^2)(1 - e_1 \mathfrak{A}_4) \qquad (31)$$

are rather large, $\mathfrak{A}_3 \simeq \alpha_s^3\left(1 - 2(\pi\beta_0)^2\alpha_s^2\right), \mathfrak{A}_4 \simeq \alpha_s^4\left[1 - (10/3)(\pi\beta_0)^2\alpha_s^2\right]$ in the region with $\pi^2\beta_0^2(5) = 3.57$ and $\alpha_s(\xi M_\Upsilon^2) \simeq 0.2$. As a crude estimate (taken from $\alpha_s^3 \to \mathfrak{A}_3$ only),

$$\Delta\alpha_s(M_\Upsilon^2) = \frac{2}{3}(\pi\beta_0)^2\alpha_s^3(M_\Upsilon^2) \simeq 0.0123,$$

which corresponds to

$$\Delta\bar{\alpha}_s(M_Z^2) = 0.006, \quad \text{with result } \bar{\alpha}_s(M_Z^2) = 0.120. \qquad (32)$$

One should note here that this estimate is rather crude and gives only an indication of the order of magnitude.

[1] In particular, this relates to the analysis of τ decay. In this connection we would like to direct attention to the important paper of [33] that treats the τ decay within the APT approach (with effective mass of the light quarks and the threshold resummation factor) and results in $\Lambda^{(3)} = 420\,\mathrm{MeV}$, which corresponds to $\alpha_{an}(M_\tau^2) = 0.32$ or $\tilde{\alpha}(M_\tau^2) = 0.30$. At the same time, attempts to interpret the results of APT for τ decay in terms of α_s, like, e.g., in [35], need some special precaution; see the next footnote. A more detailed comment on the theoretical analysis of the τ decay will be published elsewhere.

[2] Generally, it is possible to use the correspondence between α_{an}, $\tilde{\alpha}$ and α_s as expressed by the relations (14). However, the use of $\alpha_s^{\overline{\mathrm{MS}}}(\mu^2)$ at $\mu \lesssim 1\,\mathrm{GeV}$ as a QCD parameter could be misleading due to the vicinity to the singularity. For example, at $\Lambda^{(3)} = 400\,\mathrm{MeV}$ one has $\alpha_s(M_\tau^2) \simeq 0.35$ and $\alpha_s(1\,\mathrm{GeV}^2) \simeq 0.55$, to be compared with $\alpha_{an}(M_\tau^2) \simeq 0.31$ and $\alpha_{an}(1\,\mathrm{GeV}^2) \simeq 0.40$.

[3] See, e.g., (9.16) in [1].

[4] A first proposal of taking into account this effect in the Υ decay was discussed [10] more than a quarter of a century ago. Nevertheless, in current practice it is completely forgotten.

The NNLO case. Now, let us turn to a few cases analyzed by the three-term expansion formula (1). For the first example, take e^+e^- hadron annihilation at $s^{1/2} = 42\,\text{GeV}$ and 11 GeV.

A common form (see, e.g., (15) in [2]) of the theoretical presentation of the QCD correction in our normalization looks like

$$r_{e^+e^-}(s^{1/2}) = 0.318\bar{\alpha}_s(s) + 0.143\bar{\alpha}_s^2 - 0.413\bar{\alpha}_s^3 .$$

In the standard PT analysis, one has (see, e.g., Table 3) $\bar{\alpha}_s(42^2) = 0.144$ which corresponds to $r_{e^+e^-}(42) \simeq 0.0476$. Along with the APT prescription, one should use

$$r_{e^+e^-}(\sqrt{s}) = 0.318\tilde{\alpha}(s) + 0.143\mathfrak{A}_2(s) - 0.023\mathfrak{A}_3(s), \quad (33)$$

which yields $\tilde{\alpha}(42^2) = 0.142 \to \alpha_s(42^2) = 0.145$ and $\bar{\alpha}_s(M_Z^2) = 0.127$, to be compared with $\bar{\alpha}_s(M_Z^2) = 0.126$ of the usual analysis.

Quite analogously, with $\bar{\alpha}_s(11^2) = 0.200$ and $r_{e^+e^-}(11) \simeq 0.0661$ we obtain via (33) $\tilde{\alpha}(11^2) = 0.190$ which corresponds to $\bar{\alpha}_s(M_Z^2) = 0.129$ instead of 0.130.

For the next example, take the Z_0 inclusive decay. The observed ratio

$$R_Z = \Gamma(Z_0 \to \text{hadrons})/\Gamma(Z_0 \to \text{leptons}) = 20.783 \pm .029$$

can be written down as follows: $R_Z = R_0\left(1 + r_Z(M_Z^2)\right)$ with $R_0 = 19.93$. A common form (see, e.g., (15) in [2]) of presenting the QCD correction r_Z looks like

$$r_Z(M_Z) = 0.3326\bar{\alpha}_s + 0.0952\bar{\alpha}_s^2 - 0.483\bar{\alpha}_s^3 .$$

To $[r_Z]_{\text{obs}} = 0.04184$ there corresponds $\bar{\alpha}_s(M_Z^2) = 0.124$ with $\Lambda^{(5)}_{\overline{\text{MS}}} = 292\,\text{MeV}$. In the APT case, from

$$r_Z^{\text{APT}}(M_Z) = 0.3326\tilde{\alpha}(M_Z^2) + 0.0952\mathfrak{A}_2(M_Z^2) - 0.094\mathfrak{A}_3(M_Z^2) \quad (34)$$

we obtain $\tilde{\alpha}(M_Z^2) = 0.122$ and $\bar{\alpha}_s(M_Z^2) = 0.124$. Note that here the three-term approximation (8) gives the same relation between the $\bar{\alpha}_s(M_Z^2)$ and $\tilde{\alpha}(M_Z^2)$ values.

Nevertheless, in accordance with our preliminary estimate for the $(\triangle\bar{\alpha}_s)_4$ role, even the so-called NNLO theory needs some π^2 correction in the $W = s^{1/2} \lesssim 50\,\text{GeV}$ region.

The NLO case. Now, turn to those experiments in the HE ($f = 5$) Minkowskian region (mainly with a shape analysis) that usually are associated with the two-term expression (24). As has been shown above (27), the main theoretical error here can be expressed in the form

$$(\triangle\bar{\alpha}_s(s)|^{f=5}_{20\div 100\,\text{GeV}} \simeq 1.225\bar{\alpha}_s^3(s) \simeq 0.002 \div 0.003. \quad (35)$$

An adequate expression for the equivalent shift of the $\bar{\alpha}_s(M_Z^2)$ value is

$$[\triangle\bar{\alpha}_s(M_Z^2)]_3 = 1.225\bar{\alpha}_s(s)\bar{\alpha}_s(M_Z^2)^2. \quad (36)$$

We give the results of our approximate APT calculations, mainly by (35) and (36), in the form of Table 4 and Fig. 3. In the last column of Table 4, in brackets we indicate the difference between the APT and the usual analysis. The results of the three-loop analysis are marked by bold figures. Dots in the lower part of the table correspond to shape–events data for energies $W = 133, 161, 172$ and 183 GeV with the same positive shift 0.002 for the extracted $\bar{\alpha}_s$ values.

In Fig. 3 by open and hatched circles we give two- and three-loop data from Fig. 10 of paper [2]. The only exclusion is the Υ decay taken from Table X of the same paper. By crosses, we marked the "APT values" calculated approximately by (35).

Table 4. The APT revised[a] part ($f = 5$) of Bethke's [2] Table 6

Process	$s^{1/2}$ GeV	loops No	$\bar{\alpha}_s(s)$ [2]	$\bar{\alpha}_s(M_z^2)$ [2]	$\bar{\alpha}_s(s)$ APT	$\bar{\alpha}_s(M_z^2)$ APT
Υ decay[b]	9.5	2	.170	.114	.182	.120 (+6)
$e^+e^-[\sigma_{\text{had}}]$	10.5	**3**	.200	.130	.198	.129(−**1**)
e^+e^-[j & sh]	22.0	2	.161	.124	.166	.127(+3)
e^+e^-[j & sh]	35.0	2	.145	.123	.149	.126(+3)
$e^+e^-[\sigma_{\text{had}}]$	42.4	**3**	.144	.126	.145	.127(+**1**)
e^+e^-[j & sh]	44.0	2	.139	.123	.142	.126(+3)
e^+e^-[j & sh]	58	2	.132	.123	.135	.125(+2)
$Z_0 \to$ had.	91.2	**3**	.124	.124	.124	.124(**0**)
e^+e^-[j & sh]	91.2	2	.121	.121	.123	.123(+2)
-"-	...	2	...	...	...	...(+2)
e^+e^-[j & sh]	189	2	.110	.123	.112	.125(+2)
Averaged $\langle\bar{\alpha}_s(M_z^2)\rangle_{f=5}$ values				0.121;		0.124

[a] "j & sh" is for jets and shapes; figures in brackets in the last column give the difference $\triangle\bar{\alpha}_s(M_Z^2)$ between common and APT values
[b] Taken from [1]

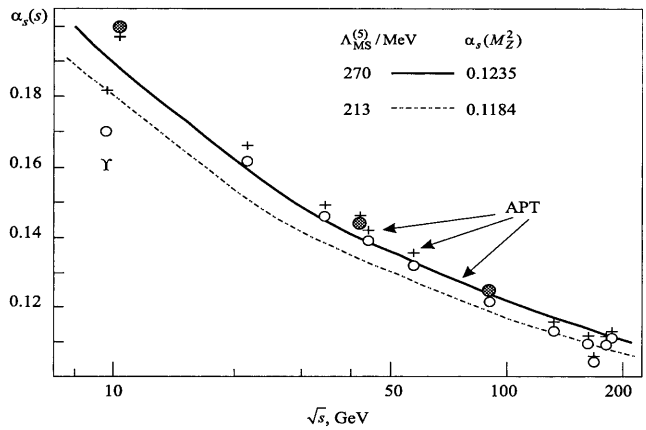

Fig. 3. The APT analysis for $\bar{\alpha}_s$ in the $f = 5$ time-like region. Crosses (+) differ from circles by the π^2 correction (35). The solid APT curve relates to $\Lambda^{(5)}_{\overline{\text{MS}}} = 270\,\text{MeV}$ and $\bar{\alpha}_s(M_Z^2) = 0.124$. By the dot-dashed curve we give the standard $\bar{\alpha}_s$ (at $\Lambda^{(5)} = 213\,\text{MeV}$ and $\bar{\alpha}_s(M_Z^2) = 0.118$) taken from Fig. 10 of [2]

For clarity of the π^2 effect, we skipped the error bars. They are the same as in the mentioned Bethke figure and we used them for calculating χ^2.

Let us note that our average 0.121 over events from Table 6 of Bethke's review [2] nicely correlates with recent data of the same author (see the summary of [36]). The best χ^2 fit yields $\bar{\alpha}_s(M_Z^2)_{[2]} = 0.1214$ and[5]

$$\bar{\alpha}_s(M_Z^2)_{\rm APT} = 0.1235.$$

This new $\chi^2_{\rm APT}$ is smaller $\chi^2_{\rm APT}/\chi^2_{\rm PT} \simeq 0.73$ than the usual one. This illustrates the effectiveness of the APT procedure in the region far enough from the ghost singularity.

5 Conclusion

It is a common standpoint that in QCD it is legitimate to use the expansion in powers of α_s for observables in the low-energy (low momentum transfer) region. At the same time, there exist rather general (and old [37]) arguments in favor of non-analyticity of the S matrix elements at the origin [38] of the complex plane of the α expansion parameter variable. This, in turn, implies that the common perturbation expansion has no domain of convergence. Technically, this corresponds to a factorial growth ($\sim n!$) of the expansion coefficients (like d_n or r_n) at large n [39, 40]. In QCD, with its "not small enough" α_s values in the region below 10 GeV it is a popular belief that one does face the asymptotic nature of the perturbation expansion by observing the approximate equality of the relative contributions of the second (α_s^2) and the third (α_s^3) terms to the observable, like in all PT columns of Table 2.

Our first qualitative result consists in the observation that the convergence properties of the APT expansions drastically differ from the usual PT ones.

The evidently better practical convergence of the APT series for the Euclidean observable, as has been demonstrated in the right part of Table 2, probably means that the essential singularity at $\alpha_s = 0$ is adequately taken into account by the new expansion functions $\mathcal{A}_k(q^2)$. On the other hand, in the time-like region the improved approximation property of the APT expansion over $\mathfrak{A}_k(s)$ has a bit different nature, being related, in our opinion, to the non-uniform convergence of the standard PT expansion for Minkowskian observables. In any case, from a practical point of view
(1) in the APT, one can use the non-power expansions (22) and (23) without the last term.

The next point, discussed in Sect. 3.3, refers to a more specific issue connected with the current practice of the Minkowskian observable analysis in the low-energy ($s \lesssim 3\,\mathrm{GeV}^2$) region (like, e.g., inclusive τ decay). As has been shown (see Fig. 3.3)
(2) below 2 GeV^2 it is impossible to use the common power expansion (1) for a time-like observable.

A second group of results is of a quantitative nature. They are
(3) the effective positive shift $\Delta\bar{\alpha}_s \simeq +0.002$ in the upper half (≥ 50 GeV) of the $f = 5$ region for all time-like events that have been analyzed up to now in the NLO mode;
(4) an effective shift $\Delta\bar{\alpha}_s \gtrsim +0.003$ in the lower half ($10 \div 50$ GeV) of the $f = 5$ region for all time-like events that have been analyzed in the NLO mode;
(5) the new value

$$\bar{\alpha}_s(M_Z^2) = 0.124, \qquad (37)$$

obtained by averaging new APT results over the $f = 5$ region.

These quantitative results are based on the new APT non-power expansion (8) and the plausible hypothesis on the π^2 term prevalence in common expansion coefficients for observables in the Minkowskian domain. The hypothesis has some preliminary support (see Table 1) but needs to be checked in more detail.

Nevertheless, our result (37) being taken for granted raises two physical questions:
(1) the issue of the self-consistency of the QCD invariant coupling behavior between the "medium ($f = 3, 4$)" and "high ($f = 5, 6$)" regions.

Here, more detailed APT analyses of the data on DIS, heavy quarkonium decays and some other processes are in order. As has been mentioned above, a fresh APT analysis of the τ decay [33] seems to support such a correlation with $\Lambda_3 \sim 400 \div 450$ MeV and $\Lambda_5 \sim 290$ MeV.
(2) The new "enlarged value" (37) can influence various physical speculations in the very HE region, in particular concerning the superpartner masses in MSSM GUT constructions – compare, e.g., with recent attempts [41] in this direction.

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[5] This value, corresponding to $\Lambda^{(5)} = 290$ MeV, is supported by a recent analysis [33] of the τ decay that gives $\Lambda^{(3)} = 420$ MeV; compare with Table 3.

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